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Some Applications in Economy for Utility Functions Involving Risk Theory

Corina Cipu[a], Carmen Gheorghe[b]*

[a]University Politehnica of Bucharest, Romania, Str. Polizu, no. 1-7, Sector 1, Bucharest, 011061, Romania
[b]National Institute for Economic Research "Costin C. Kiriţescu", Casa Academiei Române, Calea 13 Septembrie, no. 13, Sector 5, Bucharest, Romania

Abstract

We present in the first part of the article types of utility functions that can describe the behavior of the investor and their applications to optimize portfolio. The second part of the paper refers to applications in calculating insurance premiums aggregated risk in zero utility principle.

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1. Maximize the value of the business for Risk Averse Investor

There are three important principles of investments in order to maximize the value business. These are: the investment decision, the financing decision and the dividend decision. The investment decision can be taken by investing in assets that gain a higher return than the smallest acceptable hurdle rate. The hurdle rate should reflect risk degree of the investment and the mixture of debt and equity employed to finance it, and the return should reflect the magnitude and phasing of cash payments.

* Corresponding author.
E-mail address: corinac@math.pub.ro, carmen.adriana@ince.ro;

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In order to find the most suitable type of debt and the correct blend of debt and equity to finance our business operations, we have to apply the financing decision. It is very important to match the correct type of tenor for your assets to maximize the firm value. With the dividend decision can be stated the sum to return to the business investors in cash or buybacks.

The utility functions allow the measurement of the preferences of an investor in the desire to increase wealth in view of its risk aversion. We present here several criteria for the selection of a utility function that evolved over time. Among them are:

[1] $u(\cdot)$ is an increasing function for $x \in (0,\infty)$. This is true when the first derivative (marginal utility) is strictly positive;

[2] $u(\cdot)$ is concave;

[3] $u(\cdot)$ is bounded: there exists $m \in R$ such that $u(x) < m$. It is important to isolate the large values that occur rarely from main preferences;

[4] as wealth increases, the absolute risk aversion $A(\cdot)$ decreases.

A fifth criterion is occasionally advanced:

[5] utility is a constant function for negative values of wealth $x < 0, u'(x) = 0$.

The utility functions we use in this paper are continuous and differentiable at zero, and they have the properties of

i. normalization $u'(0) = 1, u(0) = 0$ (empty financial budget has no utility);

ii. monotonicity $y, z \in R\ y \le z \to u(y) \le u(z)$ and
concavity $y, z \in R$, 0<m>1, u(my+(1-m)z) $\ge mu(y) + (1-m)u(z)$ (relative increase of the utility gets smaller when y grows).

Functions that do not meet the above criteria are $u(x) = x$ (not concave); $u(x) = 1 - e^{-\lambda x}, \lambda > 0$ (not bounded); $u(x) = x^{\lambda}, 0 < \lambda < 1$ (fails [4]). Utility functions obtained from Weibull and Pareto distribution functions that do meet the above criteria for proper parameters $u(x) = 1 - e^{-\lambda x^{a}}, a < 1, \lambda > 0$, $u(x) = 1 - (\lambda x + 1)^{-a}, a, \lambda > 0$.

Remark: The normalization conditions could always be fulfilled for utility functions with $u' > 0$ if we consider a change function $x \to \left(u(x) - \dfrac{u(0)}{u'(0)} \right)$.

If *u* is twice differentiable we can write the [1]-[3] properties $u' \ge 0, u'' \le 0, u(0) = 0$ and $u'(0) = 1$.

2. Principle of expected utility maximization

For F - feasible investment alternatives, $X(I)$ - random variable giving the ending value of the investment for the time period considered, a rational investor acts to select an investment $I_{opt} \in F$ which maximizes his expected utility function u

$$E(u\left(X\left(I_{opt}\right)\right) = \max_{I \in F}(E(X(I))$$

3. Investment problem

We take X as a random variable; x_1 and x_2 two realizations; x_1 represents good outcome and x_2 bad outcome. The set of feasible investment alternatives has only two elements: $p = P(X = x_1)$, $1 - p = P(X = x_2)$.
Question: Which alternative (do nothing or make investment) does the investor choose if he follows the principle of expected utility maximization?

The Certainty Equivalent ($Cert.Eq$) for X is $c = u^{-1}\left(M(u(X)\right)$ i.e. $u(c) = M(u(X))$, meaning: if his current wealth is c, he will be indifferent between undertaking the investment and doing nothing. For investment problem $Cert.Eq = px_1 + (1-p)x_2$.

One can measure the level of the risk aversion of an investor, in two ways:

a) measure of the absolute risk aversion by: $R_a(x) = A(x) = -\frac{u''(x)}{u'(x)}$:

- For the increasing absolute risk aversion behaviour: $A'(x) > 0 \Leftrightarrow u'''(x)u'(x) < u''(x)^2$ then as wealth increases will be hold fewer Euro in risky assets.
- For the stationary case $A'(x) = 0 \Leftrightarrow u'''(x)u'(x) = u''(x)^2$
- For the decreasing absolute risk aversion behaviour: $A'(x) < 0$ then as wealth increases will be hold more Euro in risky assets.

b) measure of the relative risk aversion: $R_r(x) = xA(x) = -\frac{x\,u''(x)}{u'(x)}$, with $A'(x) = -\frac{u'''(x)u'(x) - u''(x)^2}{\left(u'(x)\right)^2}$.

4. Application in insurances. Utility premiums for linear truncated and quadratic utility

Compensations are paid by IC (insurance company) based on insurance policies owned by the insured following the conclusion of a contract. Premiums paid by insurers must cover any damages and other costs: fees, taxes, maintenance costs. The amount of damage or loss associated with a contract for a period of time is a random variable X and represents the risk assumed by IC.

The zero utility principle for different scale invariant utility functions

$$u_\lambda(x) = \frac{u(g(\lambda)x)}{g(\lambda)}, \lambda > 0, x \in \mathbb{R}, \tag{4.1}$$

with $u: R \to R$ a classical utility function, will be used in order to obtain the zero utility premium $H_{\lambda,1}$ that is the implicit solution of the following equation:

$$E(\mu_\lambda(H - Y)) = 0, \tag{4.2}$$

Y being the risk assured, with unit expectation.

For the scaled risk $X = \mu Y$ with μ expectation the zero utility premium $H_{\lambda,\mu}$ is also uniquely determined:

$$H_{\lambda,\mu} = \mu H_{\lambda/\mu,1} \tag{4.3}$$

We shall compare the variation of the zero utility premium $H_{\lambda,\mu}$ for linear truncated and quadratic utility versus λ parameter for an Exponential and Pareto distribution of the risk with μ expectation.

The linear truncated utility $u_\lambda(x) = \min\{x, \lambda\}$ leads to the approximate solution $H_{\lambda,\mu} \approx \mu$.

For X exponential with unit expectation equation (4.1) is written:

$$E[u_\lambda(H - X)] = \lambda - e^{-(H-\lambda)}, H \ge \lambda \tag{4.4}$$

and one obtain

$$H_{\lambda,1} = \begin{cases} \lambda - \ln\lambda, & \lambda \le 1 \\ 1, & \lambda \ge 1. \end{cases} \tag{4.5}$$

which is generalized for the risk X with μ expectation

$$H_{\lambda,\mu} = \begin{cases} \mu, & 0 \le \mu \le \lambda \\ \lambda + \mu \ln\left(\frac{\mu}{\lambda}\right), & \mu \ge \lambda. \end{cases} \tag{4.6}$$

Also, the mean value of $u_\lambda(H-Y)$ for the Pareto risk Y with unit expectation and α parameter is

$$E[u_\lambda(H-Y)] = \begin{cases} \lambda - \left(1 + \dfrac{H-\lambda}{\alpha}\right)^{-\alpha}, & H \geq \lambda \\ H-1, & H \leq \lambda \end{cases} \tag{4.7}$$

leading to

$$H_{\lambda,1} = \begin{cases} \lambda - 1 + \dfrac{\alpha}{\sqrt[\alpha]{\lambda}}, & \lambda \leq 1 \\ 1, & \lambda \geq 1. \end{cases} \text{ and } H_{\lambda,\mu} = \begin{cases} \mu, & 0 \leq \mu \leq \lambda \\ \lambda - a\mu + \alpha\mu\sqrt[\alpha]{\dfrac{\mu}{\lambda}}, & \mu \geq \lambda. \end{cases} \tag{4.8}$$

for the Pareto risk X with μ expectation and α parameter.

In the same manner, for quadratic utility

$$u_\lambda(x) = \begin{cases} x - \dfrac{x^2}{2\lambda}, & x \leq \lambda \\ \dfrac{\lambda}{2}, & x \geq \lambda, \end{cases} \tag{4.9}$$

the approximate solution is

$$H_{\lambda,\mu} \approx \mu + \frac{\mu^2}{2\lambda} \tag{4.10}$$

for any type of risk.
For the exponential risk Y with unit expectation the mean of $u_\lambda(H-Y)$ is

$$E[u_\lambda(H-Y)] = \begin{cases} \dfrac{\lambda^2 - 2e^{-(H-\lambda)}}{2\lambda}, & H \geq \lambda \\ \dfrac{(\lambda^2-1) - (H-(\lambda+1))^2}{2\lambda}, & H \leq \lambda \end{cases} \tag{4.11}$$

with unique solution

$$H_{\lambda,1} = \begin{cases} \lambda + 1 - \sqrt{\lambda^2 - 1} & \lambda \geq \sqrt{2} \\ \lambda - \ln\dfrac{\lambda^2}{2}, & \lambda \leq \sqrt{2}. \end{cases} \tag{4.12}$$

and

$$H_{\lambda,\mu} = \begin{cases} \lambda + \mu - \sqrt{\lambda^2 - \mu^2}, & 0 \leq \mu \leq \lambda/\sqrt{2} \\ \lambda + \mu \ln\dfrac{2\mu^2}{\lambda^2}, & \mu \geq \lambda/\sqrt{2}. \end{cases} \tag{4.13}$$

for the exponential risk X with μ expectation.

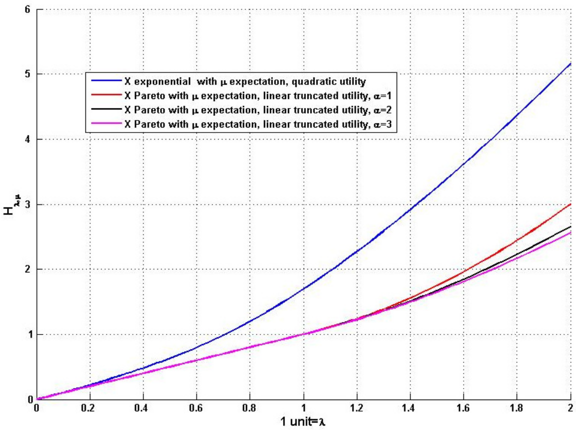

Fig 1. Zero utility premiums for exponential and Pareto risks

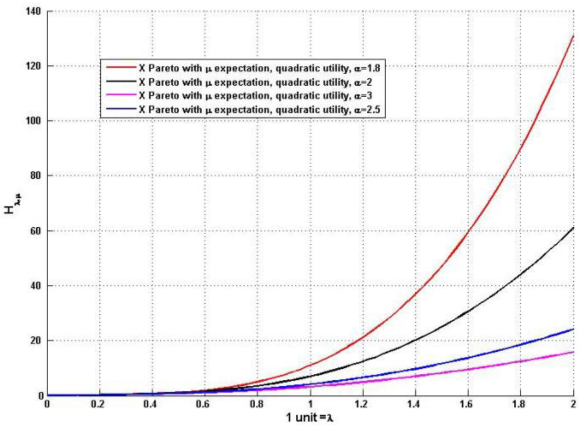

Fig 2. Quadratic utility and a Pareto risk

In Fig. 1 zero utility premiums for exponential and Pareto risks with μ expectation were represented.

One observe that between Pareto cases the Pareto for bigger α parameter is more convenient to the investor and more appropriate with the approximate solution. For quadratic utility and a Pareto risk Y with unit expectation and parameter α, the solution of equation (4.1) becomes

$$H_{\lambda,1} = \begin{cases} \lambda - \alpha + \dfrac{\alpha}{\sqrt[\alpha-1]{\dfrac{\alpha-1}{2\alpha}\lambda^2}}, & \lambda \le \sqrt{\dfrac{2\alpha}{\alpha-1}} \\ \lambda - \ln\dfrac{\lambda^2}{2}, & \lambda \ge \sqrt{\dfrac{2\alpha}{\alpha-1}}. \end{cases} \tag{4.15}$$

and for Pareto scaled risk $X = \mu Y$ with μ expectation and parameter α, the zero utility premium is

$$H_{\lambda,\mu} = \begin{cases} \lambda - \alpha\mu + \alpha\mu \sqrt[\alpha-1]{\dfrac{2\alpha}{\alpha-1}\left(\dfrac{\mu}{\lambda}\right)^2}, & \mu \ge \lambda\sqrt{\dfrac{\alpha-1}{2\alpha}} \\ \lambda + \mu - \sqrt{\lambda^2 - \dfrac{\alpha+1}{\alpha-1}\mu^2}, & 0 \le \mu \le \lambda\sqrt{\dfrac{\alpha-1}{2\alpha}}, \end{cases} \tag{4.16}$$

and was represented in Figure 2 for $\dfrac{\mu}{\lambda} \in [0,2]$ and different values of parameter $\alpha \in \{1.8; 2; 2.5; 3\}$.

When α increases to infinity $H_{\lambda,\mu}$ leads to the linearized solution.

In the case of the exponential utility function $u_\gamma(x) = \frac{1}{\gamma}(1 - e^{-\gamma x}), x \in R$ the approximate solution $H = \dfrac{1}{\lambda}\ln\dfrac{\lambda}{\lambda - y}, y < \lambda$ is not the same for any risk.

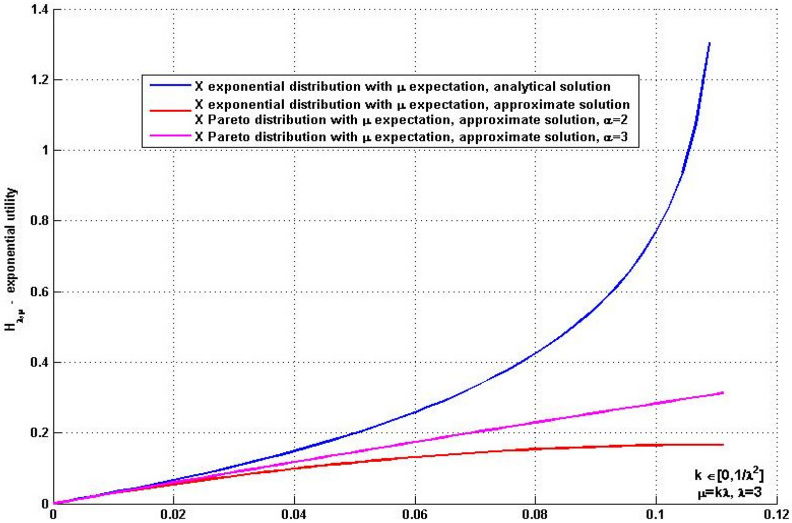

Fig.3. Exponential utility

In Figure 3 we have made comparisons between analytical an approximate solution for an exponential risk and a Pareto risk X with μ expectation, for $\lambda = 3$ and $\frac{\mu}{\lambda} \in \left[0, \frac{1}{\lambda^2}\right]$. In this case the premium functions are nonlinear in parameter λ.

Remark: For quadratic utility $A'(x) = \begin{cases} \frac{1}{(\lambda-x)^2}, x < \lambda \\ 0, x > \lambda \end{cases}$ absolute risk aversion of the investor and $R'(x) = \frac{\lambda}{(\lambda-x)^2}$ and for exponential utility $A'(x) = 0$ and $R'(x) = A(x) = \lambda$. For a quadratic utility $A'(x) > 0$ meaning absolute risk aversion of an investor and for an exponential the investor is indifferent.

5. Conclusions

According to behaviour at risk for an investor: risk averse, risk neutral or lover, and in relation to the economic problem in question, we can determine the optimal portfolio and the decision to be taken based on it and on the utility functions. For iso-elastic utility and integral type functions, considering parameters gain and percentage loss in relation to the type of event, resulting in good or bad – we have determined the optimal portfolio and the investor behaviour depending on its initial wealth. In the insurance market based zero utility principle and utility functions could be determined premiums that must be paid following an insurance risk type.

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